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Abstract. The abstract Cauchy problem for the space-time fractional evolution equation is considered, which contains the Caputo time-derivative of order $\beta \in (0,1)$ and the operator $-A^{\alpha}$, $\alpha \in (0,1)$, where $-A$ generates a strongly continuous one-parameter semigroup on a Banach space. The analyticity of the solution operator is studied by applying subordination principles for space- and time-fractional evolution equations and taking into account the asymptotic behavior of the subordination kernels, expressed in terms of Lévy extremal stable densities and Mainardi function.

INTRODUCTION

Fractional Calculus attracted the attention of many researchers in the last decades [1]. In particular, evolution equations containing both time- and space-fractional differential operators have found numerous applications and are extensively studied, see e.g. [2, 3].

This work is concerned with the abstract Cauchy problem for the space-time fractional evolution equation

$$D_t^{\beta} u(t) = -A^{\alpha} u(t),\ t > 0,\ \ u(0) = v \in X, \quad 0 < \alpha, \beta < 1. \tag{1}$$

Here D_t^{β} denotes the Caputo time-fractional derivative of order β

$$D_t^{\beta} u(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{u(t) - u(0)}{(t-\tau)^{\beta}}\, d\tau,\ \ t > 0, \quad 0 < \beta < 1, \tag{2}$$

the operator $-A$ is a generator of a bounded C_0-semigroup in a Banach space X and A^{α} denotes the α-th fractional power according to the Balakrishnan definition [4, 5]

$$A^{\alpha} v = \frac{\sin \alpha\pi}{\pi} \int_0^{\infty} \lambda^{\alpha-1} (\lambda + A)^{-1} A v \, d\lambda,\ \ v \in D(A), \quad 0 < \alpha < 1. \tag{3}$$

For the semigroup theory we refer to [5, 6], for basic theory of fractional evolution equations see e.g. [7].

Analyticity of solution operators is relevant in the study of evolution equations. For fractional order equations such a property can be derived applying the subordination principle [7, 8, 9, 10, 11], which provides an integral representation of the solution operator in terms of a scalar subordination kernel and a known solution operator, usually a C_0-semigroup.

Let us denote by $S(t)$ the C_0-semigroup generated by the operator $-A$ and by $S_{\alpha,\beta}(t)$ the solution operator of problem (1). In the limiting case $\alpha = \beta = 1$ it holds $S_{1,1}(t) = S(t)$. Subordination principle for problem (1) gives the following integral representation for the solution operator $S_{\alpha,\beta}(t)$ (see [11])

$$S_{\alpha,\beta}(t) = \int_0^{\infty} \psi_{\alpha,\beta}(t,\tau) S(\tau)\, d\tau,\ \ t > 0, \tag{4}$$

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where $\psi_{\alpha,\beta}(t,\tau)$ is a probability density function (pdf), i.e.

$$\psi_{\alpha,\beta}(t,\tau) \geq 0, \quad \int_0^\infty \psi_{\alpha,\beta}(t,\tau)\, d\tau = 1. \tag{5}$$

In [5] a subordination formula for the space-fractional version of problem (1) is established, which implies that $S_{\alpha,1}(t)$ is a bounded analytic C_0-semigroup. Concerning time-fractional evolution equations, the subordination principle is studied in [7] and extended to equations with more general time-fractional operators in [8, 9, 12, 13, 14]. Subordination principle in the setting of abstract Volterra equations is studied in [15], Chapter 4. In [10] a generalized subordination principle for problem (1) is discussed, where $-A$ is a generator of a fractional resolvent family. In [11] a subordination relation for the space-time fractional evolution equation (1) is employed for finding the sector in the complex plane, in which $S_{\alpha,\beta}(t)$ is a bounded analytic solution operator, provided the operator $-A$ generates a bounded C_0-semigroup.

In the applications, often the operator $-A$ generates a bounded analytic C_0-semigroup, for instance such is the Laplace operator, $-A = \Delta$, in $\mathbb{R}^n$, see e.g. [6]. This case is considered in the present work. Assuming that $-A$ generates a bounded analytic C_0-semigroup, the sector in the complex plane is found, in which $S_{\alpha,\beta}(t)$ is a bounded analytic solution operator. Our tools are the subordination principles for space- and time-fractional evolution equations and some asymptotic properties of the Mittag-Leffler function, the Mainardi function and the Lévy extremal stable density.

The paper is organized as follows. The next section contains preliminary results concerning the Mittag-Leffler function, the Mainardi function and the Lévy extremal stable density. The main result on the analyticity of the solution operator is formulated and proven in the third section.

MAINARDI FUNCTION AND LÉVY EXTREMAL STABLE DENSITY

Let us start with some preliminaries concerning the basic functions, which appear in our considerations: the Mainardi function and the Lévy extremal stable density. We focus on their asymptotic behavior in the complex plane.

We denote by $\Sigma(\theta)$ the sector in $\mathbb{C}$

$$\Sigma(\theta) = \{z \in \mathbb{C}\backslash\{0\};\, |\arg z| < \theta\}, \quad \theta \in [0,\pi).$$

The following notation for the Laplace transform of a function is used

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The Mainardi function $M_\gamma(\cdot)$, $0 < \gamma < 1$, can be defined by the Laplace transform pair [16, 17]

$$\mathcal{L}\{M_\gamma(\cdot)\}(s) = E_\gamma(-s), \tag{6}$$

where $E_\gamma(\cdot)$ denotes the Mittag-Leffler function, which is an entire function with the series representation [16, 17, 18]

$$E_\gamma(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\gamma n + 1)}, \quad z \in \mathbb{C}, \ \gamma > 0. \tag{7}$$

For $0 < \gamma < 2$ the following asymptotic expansion for large $|z|$ holds true

$$E_\gamma(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1-\gamma n)} + O(|z|^{-N}), \quad |\arg(-z)| < (1-\gamma/2)\pi, \quad |z| \to \infty. \tag{8}$$

The Mainardi function is an entire function of Wright type which is represented by the series [16, 17]

$$M_\gamma(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1. \tag{9}$$

It is proven in the original paper [19] that the function $M_\gamma(z)$ admits the following asymptotic expansion in the sector $|\arg z| < \min\{(1-\gamma)3\pi/2, \pi\}$

$$M_\gamma(z) \approx a(\gamma) z^{\frac{\gamma-1/2}{1-\gamma}} \exp\left(-b(\gamma) z^{\frac{1}{1-\gamma}}\right), \quad |z| \to \infty, \tag{10}$$

where $a(\gamma)$ and $b(\gamma)$ are positive constants depending only on γ, $a(\gamma) = Ab(\gamma)^{\gamma-1/2}$, $b(\gamma) = (1-\gamma)\gamma^{\gamma/(1-\gamma)}$, $A > 0$.

The Mittag-Leffler and Mainardi functions play an important role in the study of time-fractional partial differential equations, see [16, 17]. For further details on these functions and their generalizations we refer to [16, 17, 20, 21].

The Lévy extremal stable density $L_\gamma(\cdot)$ is defined by the Laplace transform pair (see e.g. [22, 23, 3])

$$\mathcal{L}\{L_\gamma(\cdot)\}(s) = \exp(-s^\gamma), \quad 0 < \gamma < 1. \tag{11}$$

It admits the series representation, see e.g. [3, 24]

$$L_\gamma(z) = \frac{1}{\pi z} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(\gamma n + 1)}{n!} \frac{\sin(\gamma n \pi)}{z^{\gamma n}}. \tag{12}$$

The first term of the series in (12) provides the following asymptotic expression of $L_\gamma(z)$ for large $|z|$ in the complex plane cut along the negative real axis

$$L_\gamma(z) \approx \frac{c(\gamma)}{z^{\gamma+1}}, \quad |z| \to \infty, \tag{13}$$

where $c(\gamma) = \frac{\sin(\gamma\pi)}{\pi}\Gamma(\gamma+1)$. By using the properties of the Gamma function $\frac{\pi}{\sin(\gamma\pi)} = \Gamma(\gamma)\Gamma(1-\gamma)$ and $\Gamma(\gamma+1) = \gamma\Gamma(\gamma)$ the expression for the constant $c(\gamma)$ simplifies to $c(\gamma) = \gamma/\Gamma(1-\gamma)$. The Lévy extremal stable density function $L_\gamma(\cdot)$ plays a crucial role in the study of space-fractional partial differential equations, see e.g. [3, 5].

It is remarkable that the functions $M_\gamma(z)$ and $L_\gamma(z)$ are related via the identity (see e.g. [16])

$$L_\gamma(z) = \gamma z^{-\gamma-1} M_\gamma\left(z^{-\gamma}\right). \tag{14}$$

This relation can be deduced for example from the series expansions (9) and (12).

Applying (14), the asymptotic behavior of $L_\gamma(z)$ for small $|z|$ can be deduced from (10)

$$L_\gamma(z) \approx \gamma a(\gamma) z^{-\frac{2-\gamma}{2(1-\gamma)}} \exp\left(-b(\gamma) z^{-\frac{\gamma}{1-\gamma}}\right), \quad |z| \to 0, \tag{15}$$

for z belonging to the sector $|\arg z| < \min\{(1/\gamma - 1)3\pi/2, \pi\}$. We notice that, by restricting z to the real positive half-line $z = t \in (0, \infty)$ in (15), resp. (13), we recover the asymptotic formulae established in [22], eqs. (4) and (5).

The sectors in the complex plane, in which $M_\gamma(z)$ and $L_\gamma(z)$ are bounded analytic functions can be easily found from the behavior of their Laplace transforms (6) and (11), applying the following statement ([15], Theorem 0.1.):

If $G(\cdot)$ is a function defined on $(0, \infty)$ and $\theta_0 \in (0, \pi/2]$ then the assertions (i) and (ii) are equivalent:

(i) $G(s)$ admits analytic extension to the sector $\Sigma(\pi/2 + \theta_0)$ and $sG(s)$ is bounded on the sector $\overline{\Sigma(\pi/2 + \theta)}$ for each $\theta < \theta_0$;

(ii) there is a function $g(t)$ analytic in $\Sigma(\theta_0)$ and bounded on each proper subsector $\overline{\Sigma(\theta)}$, $\theta < \theta_0$, such that $G(s) = \mathcal{L}\{g(\cdot)\}(s)$ for each $s > 0$.

The asymptotic expansion for the Mittag-Leffler function (8) implies that $s\mathcal{L}\{M_\gamma(\cdot)\}(s) = sE_\gamma(-s)$ is bounded for $s \in \Sigma((1-\gamma/2)\pi)$. On the other hand, $s\mathcal{L}\{L_\gamma(\cdot)\}(s) = s\exp(-s^\gamma)$ is bounded when $|\arg(s^\gamma)| < \pi/2$, i.e. for $s \in \Sigma(\gamma^{-1}\pi/2)$. Based on the above statement, we deduce that the functions $M_\gamma(z)$ and $L_\gamma(z)$ are analytic in the sectors $\Sigma(\theta_M)$ and $\Sigma(\theta_L)$, respectively, and bounded on each proper subsector, where

$$\theta_M(\gamma) = (1-\gamma)\pi/2 \ \text{ for } \ M_\gamma(z); \qquad \theta_L(\gamma) = (1/\gamma - 1)\pi/2 \ \text{ for } \ L_\gamma(z). \tag{16}$$

In fact, the asymptotic expansions for the functions M_γ and L_γ imply a stronger property in the same sectors as we see next. Concerning M_γ, expansion (10) implies that there exists r^*, such that for any $r > r^*$

$$|M_\gamma(re^{i\theta}))| \leq a(\gamma) r^{\frac{\gamma-1/2}{1-\gamma}} \exp\left(-b(\gamma) r^{\frac{1}{1-\gamma}} \cos\left(\frac{\theta}{1-\gamma}\right)\right).$$

Therefore, this function is integrable at $r \to \infty$ provided $|\theta| < (1-\gamma)\pi/2 = \theta_M(\gamma)$, recovering the same angle as in (16). In addition, (9) shows that $|M_\gamma(re^{i\theta})|$ is a bounded function as $r \to 0$. Therefore, the following integral is uniformly bounded

$$\int_0^\infty |M_\gamma(re^{i\theta})|\, dr \le C_M, \quad |\theta| < \theta_M(\gamma). \tag{17}$$

Concerning the function L_γ, expansion (13) shows that $|L_\gamma(re^{i\theta})|$ admits an integrable singularity for $r \to \infty$. For small r the asymptotic expression (15) implies the estimate

$$|L_\gamma(re^{i\theta})| \le \gamma a(\gamma) r^{-\frac{2-\gamma}{2(1-\gamma)}} \exp\left(-b(\gamma) r^{-\frac{\gamma}{1-\gamma}} \cos\left(\frac{\gamma\theta}{1-\gamma}\right)\right).$$

This shows that the function $|L_\gamma(re^{i\theta})|$ is integrable for $r \to 0$ provided $|\theta| < (1/\gamma - 1)\pi/2 = \theta_L(\gamma)$, the same angle as defined in (16). Therefore, we established the uniform boundedness of the integral

$$\int_0^\infty |L_\gamma(re^{i\theta})|\, dr \le C_L, \quad |\theta| < \theta_L(\gamma). \tag{18}$$

The constants C_M and C_L in (17) and (18) depend on γ, but do not depend on θ.

In this way we obtained the sectors $\Sigma(\theta_M(\gamma))$ and $\Sigma(\theta_L(\gamma))$ of "good behavior" of the functions $M_\gamma(z)$ and $L_\gamma(z)$, respectively, and are ready to proceed with the study of the analyticity of the solution operator by using the subordination principle.

SUBORDINATION RELATIONS AND THE MAIN RESULT

Consider a Banach space X with norm $\|.\|$ and a closed linear operator A with dense domain $D(A) \subset X$. Assume the operator $-A$ is the infinitesimal generator of a bounded C_0-semigroup $S(t)$. Therefore, A is a non-negative operator, i.e. $(-\infty, 0) \subset \varrho(A)$ - the resolvent set of A, and

$$\|\lambda(\lambda + A)^{-1}\| < M < \infty, \quad \lambda > 0.$$

Then for $0 < \alpha < 1$ the fractional power A^α of the non-negative operator A can be defined using the Balakrishnan definition (3), which is employed in the present work.

Examples of operators $-A$ satisfying the above assumptions are the Laplace operator in different settings [3, 25], more general elliptic operators [6], second-order differential operators with non-local boundary conditions [26], etc.

It is known (see [5], Chapter IX) that $-A^\alpha$ is a closed densely defined operator, which generates a bounded analytic C_0-semigroup $S_{\alpha,1}(t)$, related to the original semigroup $S(t)$ via the identity

$$S_{\alpha,1}(t) = \int_0^\infty f_\alpha(t,\tau) S(\tau)\, d\tau, \quad t > 0. \tag{19}$$

The subordination kernel $f_\alpha(t,\tau)$ in (19) is represented in terms of the Lévy extremal stable density function $L_\alpha(\cdot)$ defined in (11) as follows, see e.g. [3, 11]

$$f_\alpha(t,\tau) = t^{-1/\alpha} L_\alpha(\tau t^{-1/\alpha}). \tag{20}$$

Since $-A^\alpha$ is a generator of a C_0-semigroup $S_{\alpha,1}(t)$, the Cauchy problem for the fractional evolution equation (1) is well posed and the solution operator admits the representation ([7], Theorem 3.1)

$$S_{\alpha,\beta}(t) = \int_0^\infty \varphi_\beta(t,\tau) S_{\alpha,1}(\tau)\, d\tau, \quad t > 0, \tag{21}$$

where

$$\varphi_\beta(t,\tau) = t^{-\beta} M_\beta(\tau t^{-\beta}) \tag{22}$$

with $M_\beta(\cdot)$ being the Mainardi function (6).

The following definition is basic in the present work.

A solution operator $S(t)$ is said to be a bounded analytic solution operator of angle $\theta_0 \in (0, \pi/2]$ if $S(t)$ admits an analytic extension to the sector $\Sigma(\theta_0)$, which is bounded on $\overline{\Sigma(\theta)}$ for each $\theta \in (0, \theta_0)$.

This definition is an extension of the definition of bounded analytic semigroup, and some caution is appropriate (see e.g. [6], Def. 3.7.3): a solution operator $S(t)$, which is bounded (i.e. bounded for $t \in (0, \infty)$), and admits an analytic extension to some sector in the complex plane, is not necessarily a bounded analytic solution operator.

In [11] we have proved that if $0 < \alpha, \beta \le 1$, $\alpha + \beta < 2$, and if $-A$ is the generator of a bounded C_0-semigroup $S(t)$ on X, then problem (1) admits a bounded analytic solution operator $S_{\alpha,\beta}(t)$ of angle θ, where

$$\theta = \min\left\{\frac{(2-\alpha-\beta)\pi}{2\beta}, \frac{\pi}{2}\right\}. \tag{23}$$

Now we suppose that the C_0-semigroup generated by the operator $-A$ is a bounded analytic semigroup. In this case we expect that the solution operator $S_{\alpha,\beta}(t)$ will be bounded analytic in a larger sector of the complex plane. The precise result, which is the main result of this work, is given in the next statement:

If $0 < \alpha, \beta \le 1$, $\alpha + \beta < 2$, and if $-A$ is the generator of a bounded analytic C_0-semigroup of angle $\phi_0 \in (0, \pi/2]$, then $S_{\alpha,\beta}(t)$ is a bounded analytic solution operator of angle θ_0, where

$$\theta_0 = \min\left\{\frac{\alpha\phi_0}{\beta} + \frac{(2-\alpha-\beta)\pi}{2\beta}, \frac{\pi}{2}\right\}. \tag{24}$$

The proof of this statement is divided into two steps, based on subordination identities (19) and (21). We use the uniform boundedness of the integrals in (17) and (18), established in the previous section.

First step. Assume $S(t)$ is a bounded analytic semigroup of angle ϕ_0, i.e. $S(t)$ admits an analytic extension to the sector $\Sigma(\phi_0)$ and it is bounded on each proper subsector, i.e.

$$\|S(z)\| \le C, \quad z \in \overline{\Sigma(\phi)}, \quad \phi < \phi_0. \tag{25}$$

We start from the subordination identity (19). Let us consider the path in the complex plane

$$\Gamma_{R,\phi} = \{z = r, r \in [0, R]\} \cup \{z = Re^{i\varphi}, \varphi \in [0, \phi]\} \cup \{z = re^{i\phi}, r \in [0, R]\}, \quad \phi \in (-\phi_0, \phi_0),\ R > 0, \tag{26}$$

with counter-clockwise orientation. An application of Cauchy's theorem shows that $\int_{\Gamma_{R,\phi}} f_\alpha(t,z)S(z)\,dz = 0$ and, letting $R \to \infty$, we obtain from (19) and (20)

$$S_{\alpha,1}(t) = \int_0^\infty f_\alpha(t, re^{i\phi})S(re^{i\phi})e^{i\phi}\,dr = t^{-1/\alpha}\int_0^\infty L_\alpha(re^{i\phi}t^{-1/\alpha})S(re^{i\phi})e^{i\phi}\,dr, \quad t > 0.$$

Now let

$$S_{\alpha,1}(z) = z^{-1/\alpha}\int_0^\infty L_\alpha(re^{i\phi}z^{-1/\alpha})S(re^{i\phi})e^{i\phi}\,dr, \quad \alpha\phi - (1-\alpha)\pi/2 < \arg z < \alpha\phi + (1-\alpha)\pi/2. \tag{27}$$

Let us set $z = \rho e^{i\psi}$ with $\rho > 0$. Then (27) implies $|\phi - \psi/\alpha| < \theta_L(\alpha)$ with θ_L defined in (16) and

$$S_{\alpha,1}(z) = \rho^{-1/\alpha}e^{-i\psi/\alpha}\int_0^\infty L_\alpha(re^{i\phi}\rho^{-1/\alpha}e^{-i\psi/\alpha})S(re^{i\phi})e^{i\phi}\,dr = e^{i(\phi-\psi/\alpha)}\int_0^\infty L_\alpha(\sigma e^{i(\phi-\psi/\alpha)})S(\sigma\rho^{1/\alpha}e^{i\phi})\,d\sigma,$$

where we have set $\sigma = \rho^{-1/\alpha}r$. Therefore, applying (25) and (18) we deduce

$$\|S_{\alpha,1}(z)\| \le \int_0^\infty |L_\alpha(\sigma e^{i(\phi-\psi/\alpha)})|\,\|S(\sigma\rho^{1/\alpha}e^{i\phi})\|\,d\sigma \le C\int_0^\infty |L_\alpha(\sigma e^{i(\phi-\psi/\alpha)})|\,d\sigma \le C_1. \tag{28}$$

Varying $\phi \in (-\phi_0, \phi_0)$ in (27) provides an analytic extension of $S_{\alpha,1}$ to the sector $\Sigma(\phi_\alpha)$, which is bounded on each proper subsector, where

$$\phi_\alpha = \alpha\phi_0 + (1-\alpha)\pi/2. \tag{29}$$

Second step. Let us apply now the subordination identity (21), where $S_{\alpha,1}(t)$ is a bounded analytic solution operator of angle ϕ_α, defined in (29). We proceed in a way analogous to the first step. Take $\phi \in (-\phi_\alpha, \phi_\alpha)$ and consider the path (26). By applying the Cauchy's theorem it follows that $\int_{\Gamma_{R,\phi}} \varphi_\beta(t,z) S_{\alpha,1}(z)\, dz = 0$ for $\phi \in (-\phi_\alpha, \phi_\alpha)$. Therefore, for $R \to \infty$ we obtain from (21), taking into account (22),

$$S_{\alpha,\beta}(t) = \int_0^\infty \varphi_\beta(t, re^{i\phi}) S_{\alpha,1}(re^{i\phi}) e^{i\phi}\, dr = t^{-\beta} \int_0^\infty M_\beta(re^{i\phi} t^{-\beta}) S_{\alpha,1}(re^{i\phi}) e^{i\phi}\, dr.$$

Consider the operator-valued function

$$S_{\alpha,\beta}(z) = z^{-\beta} \int_0^\infty M_\beta(re^{i\phi} z^{-\beta}) S_{\alpha,1}(re^{i\phi}) e^{i\phi}\, dr, \quad \phi/\beta - (1/\beta - 1)\pi/2 < \arg z < \phi/\beta + (1/\beta - 1)\pi/2. \tag{30}$$

Let $z = \rho e^{i\psi}$, $\rho > 0$. Then (30) implies $|\phi - \beta\psi| < \theta_M(\beta)$ with θ_M defined in (16) and

$$S_{\alpha,\beta}(z) = \rho^{-\beta} e^{-i\beta\psi} \int_0^\infty M_\beta(re^{i\phi} \rho^{-\beta} e^{-i\beta\psi}) S_{\alpha,1}(re^{i\phi}) e^{i\phi}\, dr = e^{i(\phi-\beta\psi)} \int_0^\infty M_\beta(\sigma e^{i(\phi-\beta\psi)}) S_{\alpha,1}(\sigma\rho^\beta e^{i\phi})\, d\sigma,$$

where we have set $\sigma = \rho^{-\beta} r$. Applying (28) and (17) it follows

$$\|S_{\alpha,\beta}(z)\| \le \int_0^\infty |M_\beta(\sigma e^{i(\phi-\beta\psi)})|\, \|S_{\alpha,1}(\sigma\rho^\beta e^{i\phi})\|\, d\sigma, \le C_1 \int_0^\infty |M_\beta(\sigma e^{i(\phi-\beta\psi)})|\, d\sigma \le C_2.$$

Therefore, varying $\phi \in (-\phi_\alpha, \phi_\alpha)$ in (30) provides an analytic extension of $S_{\alpha,\beta}$ to the sector $\Sigma(\theta_0)$, which is bounded on each proper subsector, where

$$\theta_0 = \phi_\alpha/\beta + (1/\beta - 1)\pi/2. \tag{31}$$

Combining the results of the above two steps and inserting the value (29) of ϕ_α in (31), we derive the angle of analyticity (24). In this way the main statement of this work is proven.

Particular cases of this result can be found in [6], see Theorem 3.8.3, as well as in [7, 10, 11].

Let us consider an example. Assume X is one of the spaces $L^p(\mathbb{R}^n)$, $0 \le p < \infty$, or $C_0(\mathbb{R}^n)$ of continuous functions vanishing at infinity. Let A be the negative Laplace operator defined on X with maximal domain. It is known that the operator $-A$ is a generator of a bounded analytic C_0-semigroup of angle $\pi/2$ on X, the so-called Gaussian semigroup, see e.g. [6], Example 3.7.6. Plugging $\phi_0 = \pi/2$ in (24), and taking into account that $\beta < 1$, we easily get $\theta_0 = \pi/2$, i.e. the solution operator $S_{\alpha,\beta}(t)$ is a bounded analytic solution operator of angle $\pi/2$.

Let us note that, in contrast to the above example, if the original semigroup is not analytic and $\alpha + 2\beta > 2$ then the angle of analyticity of $S_{\alpha,\beta}(t)$ is strictly smaller than $\pi/2$.

CONCLUSION

In this work we found the sector in the complex plane in which the solution operator to the space-time fractional evolution equation is bounded analytic solution operator, provided the operator $-A$ generates a bounded analytic semigroup. We used subordination principles for space- and time-fractional evolution equations and the asymptotic properties of the Mainardi function and the Lévy extremal stable density.

The employed technique can be extended to equations with more general nonlocal operators in space such as those considered in [27] and operators with a general memory kernel in time as in [12]. To make such generalizations feasible a study of the asymptotic behavior of the subordination kernels in the complex plane is necessary.

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